

Application of vertex algebras to the structure theory of certain representations over the Virasoro algebra

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December 12, 2013

Abstract

In this paper we discuss the structure of the tensor product $V'_{\alpha,\beta} \otimes L(c, h)$ of irreducible module from intermediate series and irreducible highest weight module over the Virasoro algebra. We generalize Zhang's irreducibility criterion from [Zh], and show that irreducibility depends on the existence of integral roots of a certain polynomial, induced by a singular vector in the Verma module $V(c, h)$. A new type of irreducible Vir-module with infinite-dimensional weight subspaces is found. We show how the existence of intertwining operator for modules over vertex operator algebra yields reducibility of $V'_{\alpha,\beta} \otimes L(c, h)$ which is a completely new point of view to this problem. As an example, the complete structure of the tensor product with minimal models $c = -22/5$ and $c = 1/2$ is presented.

Keywords: Virasoro algebra, highest weight module, intermediate series, minimal model, vertex operator algebra, intertwining operator

AMS classification: 17B10, 17B65, 1768, 1769

1 Introduction

Along with affine Kac-Moody algebras, the Virasoro algebra plays an important role in the theory of infinite-dimensional Lie algebras. Its irreducible weight modules with finite-dimensional weight spaces were classified in [M] - every such module is either highest (or lowest) weight module, or module belonging to intermediate series. It has been shown in [MZ1] that irreducible weight module cannot have finite-dimensional and infinite-dimensional weight subspaces simultaneously. Irreducible modules with infinite-dimensional weight subspaces have been studied recently by several authors. The earliest examples were constructed by H. Zhang in [Zh]. Later, many new classes were given in [CM] and [LLZ]. All these modules are included in a huge family of irreducible weight

modules constructed in [LZ]. Also, various families of nonweight irreducible modules were studied at the same time. Most examples are various versions of Whittaker modules, and are described in a uniform way in [MZ2].

Modules with infinite-dimensional weight subspaces over some other Lie algebras were studied motivated by their connection with the theory of vertex operator algebras (VOAs) and fusion rules in conformal field theory. Such modules over affine Lie algebras were constructed in [CP], while their relation to VOA theory was discussed in [Ad1], [Ad2], and [Ad3].

One series of such representations over the Virasoro algebra is the tensor product $V'_{\alpha,\beta} \otimes L(c, h)$ of irreducible module $V'_{\alpha,\beta}$ from intermediate series and irreducible highest weight module $L(c, h)$ introduced in [Zh]. However, this module is not always irreducible. Zhang has given irreducibility criterion under the condition $\alpha \notin \beta\mathbb{Z} + \mathbb{Z}$, but this rules out many interesting examples appearing in the theory of VOA and intertwining operators.

In this paper we generalize Zhang's criterion (Theorem 3.2) and use it to obtain new series of irreducible modules. Basically, $V'_{\alpha,\beta} \otimes L(c, h)$ is irreducible if and only if the equations $x(v_{n+1} \otimes v) = v_n \otimes v$ have solutions in the universal enveloping algebra $U(\text{Vir})$ for every $n \in \mathbb{Z}$. Such a solution can exist only if there is a singular vector in the Verma module $V(c, h)$. We show that irreducibility of tensor product module depends on (non)existence of integral roots of a certain polynomial induced by that singular vector (Theorem 4.6).

We also discuss the structure of a reducible module $V'_{0,\beta} \otimes L(c, 0)$ (Theorem 4.3) and present a new type of irreducible Vir-module with infinite-dimensional weight subspaces (Theorem 4.5).

In section 5 we show strong connection between reducibility of $V'_{\alpha,\beta} \otimes L(c, h)$ and the existence of certain intertwining operators for vertex operator algebra $L(c, 0)$. In subsection 5.1 we focus on the so called minimal models. $L(c_{p,q}, 0)$ is a rational VOA, and all of its irreducible modules are known, together with fusion rules for intertwining operators between them. Using these fusion rules we prove reducibility of certain modules $V'_{\alpha,\beta} \otimes L(c, h)$. Combining these two methods, i.e. fusion rules to show reducibility and singular vectors to prove irreducibility, we give complete results for $V'_{\alpha,\beta} \otimes L(c, h)$ when $L(c, h)$ is a minimal model for $c = c_{2,5} = -22/5$, and $c = c_{3,4} = 1/2$. When these tensor product modules are reducible, we obtain irreducible minimal models as quotient modules (Propositions 5.6-5.8). Based on these examples we conjecture that analogous results hold in general, for all minimal models. In subsection 5.2 we focus on $c = 1$ and demonstrate that reducibility of $V'_{\alpha,\beta} \otimes L(1, h)$ again coincides with the existence of known intertwining operators for $h = m^2$ and $h = m^2/4$.

Remark The irreducibility problem for the tensor product has been investigated in [CGZ] at the same time as in this paper. Making use of a "shifting technique", these authors have also shown that reducibility coincides with the existence of integral roots of associated polynomial. Their elegant proof works for minimal models as well as in general case. However, our approach using fusion rules and VOA gives a better understanding of a subquotient structure.

2 Preliminaries

The Virasoro algebra Vir is a complex Lie algebra spanned by $\{C, L_i : i \in \mathbb{Z}\}$ where C is a central element, and $[L_n, L_m] = (n - m) L_{m+n} + \delta_{m, -n} \frac{n^3 - n}{12} C$. It has a natural triangular decomposition

$$\text{Vir}_+ = \bigoplus_{n>0} \mathbb{C} L_n \quad \text{Vir}_- = \bigoplus_{n>0} \mathbb{C} L_{-n} \quad \text{Vir}_0 = \mathbb{C} L_0 \oplus \mathbb{C} C$$

Let $U(\text{Vir})$ and $U(\text{Vir}_-)$ denote the universal enveloping algebras of Vir and Vir_- , respectively. The vectors $L_{i_1} \cdots L_{i_n} \mathbf{1}$, $i_1 \leq \cdots \leq i_n$, $n \in \mathbb{N}_0$ form a standard PBW basis of $U(\text{Vir})$. If we let $i_n < 0$, we obtain a basis for $U(\text{Vir}_-)$.

In the following we recall the well known facts from representation theory of the Virasoro algebra.

Verma module $V(c, h)$ with highest weight h and central charge c is a Vir -module generated by the so called highest weight vector v such that $Cv = cv$, $L_0 v = hv$ and $L_n v = 0$ for $n > 0$. It is a free $U(\text{Vir}_-)$ -module with PBW basis $\{L_{-i_n} \cdots L_{-i_1} v : i_n \geq \cdots \geq i_1 > 0, n \in \mathbb{N}_0\}$. Nonzero vector $u \in V(c, h)$ is called a singular vector if $\text{Vir}_+ u = 0$ and $L_0 u = (h + n)u$ for some $n \in \mathbb{N}_0$. Each singular vector generates a submodule of $V(c, h)$, and every proper nontrivial submodule contains at least one singular vector. Every Verma module contains proper maximal (possibly trivial) submodule $J(c, 0)$, and the quotient $L(c, h) = V(c, h)/J(c, h)$ is a unique irreducible highest weight module with the highest weight (c, h) . Each highest weight module is a quotient of the corresponding Verma module. If $J(c, h)$ is nontrivial, it is generated either by one or by two singular vectors of different weights. In this paper, we use the notation $c_{p,q} = 1 - 6 \frac{(p-q)^2}{pq}$ for $p, q > 1$ relatively prime. $J(c, h)$ is generated by two singular vectors only when $c = c_{p,q}$. We say that $V(c, h)$ is reducible with degree m if m is the smallest positive integer such that $J(c, h)$ contains a singular vector of weight $h + m$.

Let $\alpha, \beta \in \mathbb{C}$. Vir -module from intermediate series is defined by $V_{\alpha, \beta} = \sum_{m \in \mathbb{Z}} \mathbb{C} v_m$ with $L_n v_m = -(m + \alpha + \beta + n\beta) v_{m+n}$ and $Cv_m = 0$. Module $V_{\alpha, \beta}$ is reducible if and only if $\alpha \in \mathbb{Z}$ and $\beta = 0$, or $\beta = 1$. Since $V_{\alpha, \beta} \cong V_{\alpha+k, \beta}$ for all $k \in \mathbb{Z}$, we may assume $\alpha = 0$ if $\alpha \in \mathbb{Z}$. Define $V'_{0,0} := V_{0,0}/\mathbb{C} v_0$, $V'_{0,1} := \sum_{m \neq -1} \mathbb{C} v_m$, and $V'_{\alpha, \beta} := V_{\alpha, \beta}$ for all other pairs (α, β) . Then $V'_{\alpha, \beta}$ are all irreducible modules belonging to the intermediate series.

We define a module structure on tensor products $V'_{\alpha, \beta} \otimes L(c, h)$ by

$$L_n(v_k \otimes x) = (L_n v_k) \otimes x + v_k \otimes (L_n x).$$

It is easy to see that

$$(V'_{\alpha, \beta} \otimes L(c, h))_{h-\alpha-\beta+m} = \bigoplus_{n \in \mathbb{Z}_+} \mathbb{C} v_{n-m} \otimes L(c, h)_{h+n}$$

so $V'_{\alpha, \beta} \otimes L(c, h)$ has infinite-dimensional weight subspaces. Also, $V'_{\alpha, \beta} \otimes L(c, h)$ is generated by $\{v_m \otimes v : m \in \mathbb{Z}\}$ where v is the highest weight vector in $L(c, h)$.

3 Irreducibility of modules $V'_{\alpha,\beta} \otimes L(c, h)$

The following irreducibility criterion was proved in [Zh]:

Theorem 3.1 ([Zh]) *If $\alpha \notin \beta\mathbb{Z} + \mathbb{Z}$, then $V'_{\alpha,\beta} \otimes L(c, h)$ is irreducible if and only if $V'_{\alpha,\beta} \otimes L(c, h)$ is cyclic on every $v_m \otimes v$, $m \in \mathbb{Z}$, where v is the highest weight vector of $L(c, h)$.*

Here we expand Zhang's proof and eliminate restriction on α . (See also Theorem 1 of [CGZ] for a different proof of this theorem.)

Theorem 3.2 *Module $V'_{\alpha,\beta} \otimes L(c, h)$ is irreducible if and only if $V'_{\alpha,\beta} \otimes L(c, h)$ is cyclic on every $v_m \otimes v$, $m \in \mathbb{Z}$, where v is the highest weight vector of $L(c, h)$.*

Proof. The only if part is trivial. Suppose $V = V'_{\alpha,\beta} \otimes L(c, h)$ is cyclic on every $v_m \otimes v$. Let U be a nontrivial submodule in V , and let $x \in U$ be a nonzero weight $n - m$ vector:

$$x = v_{m-n} \otimes x_0 + v_{m-n+1} \otimes x_1 + \cdots + v_m \otimes x_n$$

where

$$x_j \in V(c, h)_{h+j}, j = 0, 1, \dots, n \text{ and } x_n \neq 0.$$

Using induction on n we find $v_k \otimes v \in U$ which proves irreducibility.

If $n = 0$, then $x = v_m \otimes x_0 \in U$ is a multiple of $v_m \otimes v$. Assume $n > 0$. Since L_1 and L_2 generate the algebra Vir_+ there exists $i \in \{1, 2\}$ such that $L_i x_n \neq 0$ (otherwise, x_n would be a highest weight vector, a contradiction with irreducibility of $L(c, h)$). The idea is to choose $w \in U(\text{Vir}_+)$ such that $wx \in U$ has at most n components. By inductive hypothesis there is $v_k \otimes v \in U$ for some $k \in \mathbb{Z}$, as long as $wx \neq 0$. Then we find a component of wx in $\mathbb{C}v_{m+l-i} \otimes L(c, h)_{h+n-i}$ to check whether $wx = 0$ and repeat the process if necessary.

For any $l \in \mathbb{Z}$ such that $l > n + i$, $m + \alpha + \beta + (l - i)\beta$ and $m + \alpha + \beta + l\beta$ are nonzero let

$$w = (m + \alpha + \beta + i\beta)(m + i + \alpha + \beta + (l - i)\beta)L_l + (m + \alpha + \beta + l\beta)L_{l-i}L_i.$$

It is easy to show that $wv_m = 0$. Since $l > n + i$ we have $L_l x_j = L_{l-i} x_j = 0$ for $j = 0, 1, \dots, n$, so

$$\begin{aligned} wx &= w \sum_{j=0}^n v_{m-j} \otimes x_{n-j} = \\ &= \sum_{j=1}^n wv_{m-j} \otimes x_{n-j} + (m + \alpha + \beta + l\beta) \sum_{j=0}^{n-i} L_{l-i} v_{m-j} \otimes L_i x_{n-j}. \end{aligned} \tag{1}$$

Therefore, if $wx \neq 0$ by the induction hypothesis there exists $v_k \otimes v \in U$. Suppose $wx = 0$. The component of wx in $\mathbb{C}v_{m+l-i} \otimes L(c, h)_{h+n-i}$ is

$$X_l^i = wv_{m-i} \otimes x_{n-i} + (m + \alpha + \beta + l\beta)L_{l-i}v_m \otimes L_i x_n,$$

and we have

$$(m + \alpha + \beta + l\beta) L_{l-i} v_m \otimes L_i x_n \neq 0.$$

If x_{n-i} and $L_i x_n$ are linearly independent or $x_{n-i} = 0$, then $X_l^i \neq 0$, so $wx \neq 0$. (In case $\alpha = 0$, $\beta = 1$ and $m-i = -1$ we have $X_l^i = (m+l+1) L_{l-i} v_m \otimes L_i x_n \neq 0$, and if $\alpha = \beta = m-i = 0$ we get $X_l^i = m L_{l-i} v_m \otimes L_i x_n \neq 0$.)

Suppose $0 \neq \lambda x_{n-i} = L_i x_n$ for some $\lambda \in \mathbb{C}^*$. If $X_l^i = 0$ we obtain

$$\begin{aligned} & - (m + \alpha + \beta + i\beta) (m + i + \alpha + \beta + (l-i)\beta) (m - i + \alpha + \beta + l\beta) \\ & + (m - i + \alpha + \beta + i\beta) (m + \alpha + \beta + (l-i)\beta) (m + \alpha + \beta + l\beta) + \\ & - \lambda (m + \alpha + \beta + (l-i)\beta) (m + \alpha + \beta + l\beta) = 0 \end{aligned} \quad (2)$$

for every $l > n-i$. Therefore, equation (2) holds for every $l \in \mathbb{C}$ and, in particular for $l = -\beta^{-1}(m + \alpha + \beta)$ if $\beta \neq 0$. Then

$$i^2 (m + \alpha + \beta + i\beta) (1 - \beta) = 0.$$

Since $i \neq 0$ we get $\beta = 1$ or $m + \alpha + \beta + i\beta = 0$.

$\beta = 1$ Suppose that $\beta = 1$. Then from (2) it follows¹ that $\lambda = -i$.

For $l > n+1+i$ and $m + \alpha + l + 1 \neq i$ set

$$w_1 = (m + \alpha + i + 2) L_l + L_{l-i-1} L_{i+1}.$$

Then $w_1 v_m = 0$ and

$$w_1 x = \sum_{j=1}^n w_1 v_{m-j} \otimes x_{n-j} + \sum_{j=0}^{n-i-1} L_{l-i-1} v_{m-j} \otimes L_{i+1} x_{n-j}.$$

Again, if $w_1 x \neq 0$ we can find $v_k \otimes v \in U$ by induction. The component of $w_1 x$ in $\mathbb{C} v_{m+l-i} \otimes L(c, h)_{h+n-i}$ is

$$-i (m + 1 + \alpha + l - i) v_{m+l-i} \otimes x_{n-i}$$

and since $x_{n-i} \neq 0$ we obtain $w_1 x \neq 0$.

$m + \alpha + \beta + i\beta = 0 \neq \beta$ Now assume $m + \alpha + \beta + i\beta = 0 \neq \beta$ (This case is not covered in Zhang's proof due to the condition $\alpha \notin \beta\mathbb{Z} + \mathbb{Z}$.) Then $L_i v_m = 0$, hence

$$L_i x = \sum_{j=1}^n L_i v_{m-j} \otimes x_{n-j} + \sum_{j=0}^{n-i} v_{m-j} \otimes L_i x_{n-j}.$$

If $L_i x \neq 0$ we can find $v_k \otimes v \in U$ as in previous cases. Let $L_i x = 0$. Consider the component of $L_i x$ in $\mathbb{C} v_m \otimes L(c, h)_{h+n-i}$:

$$i v_m \otimes x_{n-i} + v_m \otimes L_i x_n = 0.$$

¹In the original proof, the author makes a wrong conclusion that $\lambda = i = m + \alpha + 1$. We expand this part of the proof.

Therefore, $L_i x_n = -i x_{n-1} \neq 0$. Set

$$w_2 = ((l - 2i - 1)\beta + i + 1)L_l + (l - i)L_{l-i-1}L_{i+1}.$$

Again, since $w_2 v_m = 0$ we get

$$w_2 x = \sum_{j=1}^n w_2 v_{m-j} \otimes x_{n-j} + (l - i) \sum_{j=0}^{n-i-1} L_{l-i-1} v_{m-j} \otimes L_{i+1} x_{n-j}.$$

If $w_2 x \neq 0$ our proof is done. Now suppose $w_2 x = 0$ and consider its component in $\mathbb{C} v_{m-i} \otimes x_{n-j}$. For $l \in \mathbb{C}$ we have

$$-((l - 2i - 1)\beta + i + 1)((l - i)\beta - i) + (l - i)((l - 2i - 1)\beta - 1) = 0$$

In particular, for $l = i$ we have

$$i(i + 1)(1 - \beta) = 0,$$

so $\beta = 1$. This case is already covered.

$\beta = 0$ Now let $\beta = 0$. Then (2) becomes

$$(m + \alpha)(i(m - i + \alpha) + \lambda(m + \alpha)) = 0.$$

If $\alpha = -m \in \mathbb{Z}$ we may assume $\alpha = 0$ but then $m = 0$, so $v_0 \in V'_{0,0}$ which contradicts the definition of $V'_{0,0}$. Therefore $\lambda = -i \frac{m + \alpha - i}{m + \alpha}$ i.e.

$$L_i x_n = -i \frac{m + \alpha - i}{m + \alpha} x_{n-i}.$$

For $l > n + 2$, $m + l + \alpha \neq 0$ define

$$w_3 = (m + i + \alpha)(m + l + \alpha)L_{2l} - L_l L_{l-i} L_i.$$

Then $w_3 v_m = 0$ and if $w_3 x \neq 0$, our proof is done. The component of $w_3 x$ in $\mathbb{C} v_{m+2l-i} \otimes L(c, h)_{h+n-i}$ is

$$w_3 v_{m-i} \otimes x_{n-i} - L_l L_{l-i} v_m \otimes L_i x_n,$$

so $w_3 x = 0$ yields

$$i(m - i + \alpha)(m + i + \alpha) = 0.$$

Therefore $\alpha \in \mathbb{Z}$ and we may assume $\alpha = 0$. Then $m = -i$ since $\lambda \neq 0$. Let $w_4 = L_l + L_{l-i-1}L_{i+1}$. Then $w_4 v_{-i} = 0$,

$$w_4 x = \sum_{i=1}^n w v_{-i-j} \otimes x_{n-j} + \sum_{j=0}^{n-i-1} v_{-i-j} \otimes L_{i+1} x_{n-j}$$

and the component of $w_4 x$ in $\mathbb{C} v_{l-2i} \otimes L(c, h)_{h+n-i}$ is $2i^2 v_{l-2i} \otimes x_{n-i} \neq 0$. Once again we conclude that $w_4 x \neq 0$ and by induction we find that $v_k \otimes x \in U$.

This completes the proof. ■

From this point on, we write $U_n := U(\text{Vir})(v_n \otimes v)$. We note that from the proof of the previous theorem we have

Corollary 3.3 *Let M be a nontrivial submodule in $V'_{\alpha,\beta} \otimes L(c, h)$. Then M contains U_n for some $n \in \mathbb{Z}$.*

In order to prove irreducibility of $V'_{\alpha,\beta} \otimes L(c, h)$ it suffices to check that $U_n = U_{n+1}$ for every $n \in \mathbb{Z}$. Since

$$L_1(v_n \otimes v) = -(n + \alpha + 2\beta)v_{n+1} \otimes v$$

we get $U_n \supseteq U_{n+1}$ if $\alpha + 2\beta \notin \mathbb{Z}$. The opposite inclusion, however, requires a certain relation to hold in $L(c, h)$ i.e. the existence of a singular vector in $V(c, h)$. For example, $V'_{\alpha,\beta} \otimes V(c, h)$ is always reducible, even if $V(c, h)$ is irreducible, because $U_n \not\subseteq U_{n+1}$ for every n (Theorem 3 in [Zh]). If, on the other hand, Verma module $V(c, h)$ is reducible with degree m , then $V'_{\alpha,\beta} \otimes L(c, h)$ is irreducible whenever α is either transcendental over $\mathbb{Q}(c, h, \beta)$, or algebraic over $\mathbb{Q}(c, h, \beta)$ with degree greater than m (Theorem 5 in [Zh]). Basically, if there is a weight m singular vector in $V(c, h)$, we can prove irreducibility of $V'_{\alpha,\beta} \otimes L(c, h)$ for all pairs $(\alpha, \beta) \in \mathbb{C}^2$ except for those forming a certain algebraic curve (integral roots of a degree m polynomial). In the next section we expand this result in more details, and prove its converse.

4 Subquotient structure

First we show that highest weight modules appear naturally as subquotients in tensor products.

Lemma 4.1 *Let $M(c, h)$ be a highest weight module. If $U_{n+k} \subsetneq U_n \subseteq V'_{\alpha,\beta} \otimes M(c, h)$ for every $k \in \mathbb{N}$, then U_n/U_{n+1} is a highest weight module with the highest weight $h - \alpha - \beta - n$.*

Proof. Since

$$\begin{aligned} L_k(v_n \otimes v) &\in U_{n+k} \subseteq U_{n+1}, \quad k \in \mathbb{N} \\ L_0(v_n \otimes v) &= -(\alpha + \beta)v_n \otimes v + h(v_n \otimes v) \end{aligned}$$

we conclude that $v_n \otimes v + U_{n+1}$ is a cyclic highest weight vector with the highest weight $h - \alpha - \beta - n$. ■

The following important result generalizes the corollary following Theorem 4 in [Zh]

Proposition 4.2 *If $\alpha \notin \mathbb{Z}$, module $V'_{\alpha,\beta} \otimes L(c, 0)$ is irreducible.*

Proof. Since $L_{-1}v = 0$ in $L(c, 0)$, we have

$$\begin{aligned} L_{-1}(v_n \otimes v) &= -(n + \alpha)(v_{n-1} \otimes v), \forall n \in \mathbb{Z}, \\ L_1(v_n \otimes v) &= -(n + \alpha + 2\beta)(v_{n+1} \otimes v), \forall n \in \mathbb{Z}. \end{aligned}$$

If $n + \alpha + 2\beta \in \mathbb{Z}$ we use

$$L_{-1}L_2(v_n \otimes v) = \beta(n + 2 + \alpha)(v_{n+1} \otimes v) \neq 0$$

to prove $U_n = U_{n+1}$ for all n , thus completing the proof. ■

Now we focus on $V'_{0,\beta} \otimes L(c, 0)$. If $c \neq c_{p,q}$, $L(c, 0) = V(c, 0)/U(\text{Vir})L_{-1}v$ is a free module over algebra $U(\text{Vir}_- \setminus \{L_{-1}\})$ with standard PBW basis

$$\{L_{-i_n} \cdots L_{-i_1}v : i_n \geq \cdots \geq i_1 > 1\},$$

where v is a highest weight vector.

Let $\alpha = 0$ and $\beta \neq 0, 1$. Then the relations

$$\begin{aligned} L_{-1}(v_n \otimes v) &= -n(v_{n-1} \otimes v), \\ L_1(v_n \otimes v) &= -(n + 2\beta)(v_{n+1} \otimes v), \end{aligned}$$

or

$$L_{-1}L_2(v_n \otimes v) = (n + 2)\beta(v_{n+2} \otimes v) \neq 0$$

in case $n + 2\beta = 0$ show that

$$\cdots = U_{-2} = U_{-1} \supseteq U_0 = U_1 = \cdots$$

If $\beta = 0$ there is no v_0 in $V'_{0,0}$, so we have

$$\cdots = U_{-2} = U_{-1} \supseteq U_1 = U_2 = \cdots$$

and if $\beta = 1$ there is no v_{-1} , hence

$$\cdots = U_{-3} = U_{-2} \supseteq U_0 = U_1 = \cdots$$

Theorem 4.3 *Let $c \neq c_{p,q}$ and $\alpha \in \mathbb{Z}$. Module $V = V'_{\alpha,\beta} \otimes L(c, 0)$ is reducible and contains a nontrivial submodule U such that $V/U \cong V(c, h)$ where $h = 1 - \beta$ if $\beta \neq 1$, and $h = 1$ if $\beta = 1$.*

Proof. Since $\alpha \in \mathbb{Z}$ we can assume $\alpha = 0$. Suppose V is irreducible. Then $U_{-2} = U_1$ so there exists $x \in U(\text{Vir})$ such that $x(v_1 \otimes v) = v_{-2} \otimes v$. Recall that

$$x = \sum_{\substack{k_1, \dots, k_n \in (\mathbb{Z}_+)^n \\ n \in \mathbb{N}}} L_{-n}^{k_n} \cdots L_{-1}^{k_1} x_{k_1 \dots k_n}$$

for some homogeneous $x_{k_1 \dots k_n} \in U(\text{Vir}_+)$. Since $L_k(v_n \otimes v) = L_k v_n \otimes v$ for $k > 0$ and $L_{-1}(v_0 \otimes v) = 0$ we can write

$$\sum_{k=0}^m x_{k+2}(v_k \otimes v) = v_{-2} \otimes v$$

for some $x_j \in U(\text{Vir}_- \setminus \{L_{-1}\})_{-j}$, $m \in \mathbb{N}$. But then $v_m \otimes x_{m+2}v$ must be zero, leading to $x_{m+2}v = 0$, which is a contradiction since $L(c, 0)$ is free over $U(\text{Vir}_- \setminus \{L_{-1}\})$.

From Lemma 4.1 we know that V/U_1 is a highest weight module with highest weight $1 - \beta$ and cyclic generator $v_{-1} \otimes v$ (or highest weight module with highest weight 1 and generator $v_{-2} \otimes v$ if $\beta = 1$). Let us show this module is free over $U(\text{Vir}_-)$. For simplicity, suppose $\beta \neq 0$. Notice that

$$\begin{aligned} U_0 &= \{u(v_0 \otimes v) : u \in U(\text{Vir})\} = \\ &= \text{span}_{\mathbb{C}} \{u'(v_k \otimes v) : u' \in U(\text{Vir}_- \setminus \{L_{-1}\}), k \geq 0\} \end{aligned}$$

and each $u'(v_k \otimes v)$ contains a component $v_k \otimes u'v \neq 0$. Suppose V/U_0 is not free. Then there exists $x \in U(\text{Vir}_-)$ such that $x(v_{-1} \otimes v) \in U_0$. But this is a contradiction since $x(v_{-1} \otimes v)$ can not have a component $v_k \otimes u'v$ for $k \geq 0$. ■

Remark 4.4 For another proof of Theorem 4.3 see Remark 5.5 on pg. 17.

Theorem 4.5 Let $c \neq c_{p,q}$ and $\alpha \in \mathbb{Z}$. Then $U = U(\text{Vir})(v_1 \otimes v)$ is an irreducible submodule in $V'_{\alpha,\beta} \otimes L(c, 0)$, not isomorphic to some $V'_{\gamma,\delta} \otimes L(c, h)$.

Proof. Since $U = U_k$ for every $k \in \mathbb{N}$, irreducibility follows from Corollary 3.3.

Suppose there is a nontrivial Vir-homomorphism $\Phi : U \rightarrow V'_{\gamma,\delta} \otimes L(c, h)$ for $\gamma, \delta, h \in \mathbb{C}$. Since $\text{Supp } U_+ = -\beta + \mathbb{Z}$ and $\text{Supp } V'_{\gamma,\delta} \otimes L(c, h) = h - \gamma - \delta + \mathbb{Z}$, we have $h - \gamma - \delta + m = -\beta$ for some $m \in \mathbb{Z}$. Let $V'_{\gamma,\delta} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}w_n$ and w a

highest weight vector in $L(c, h)$.

Let $\beta \neq 0$. Then $U = U(\text{Vir})(v_0 \otimes v)$ and

$$\Phi(v_0 \otimes v) = w_0 \otimes x_0 + w_1 \otimes x_1 + \cdots + w_n \otimes x_n$$

where $x_j \in L(c, h)_j$. Since $L_{-1}(v_0 \otimes v) = 0$ we have $L_{-1}\Phi(v_0 \otimes v) = 0$, i.e.

$$\sum_{i=0}^n (i + \gamma) w_{i-1} \otimes x_i = \sum_{i=0}^n w_i \otimes L_{-1}x_i \quad (3)$$

Suppose $\gamma \in \mathbb{Z}$ i.e. $\gamma \neq 0$. Then (3) becomes

$$\sum_{i=0}^{n-1} (i + 1) w_i \otimes x_{i+1} = \sum_{i=0}^n w_i \otimes L_{-1}x_i$$

which leads to $x_{i+1} = \frac{1}{i+1} L_{-1}x_i$ for $i = 0, 1, \dots, n-1$ and $L_{-1}x_n = 0$. Hence,

$$x_j = \frac{1}{j!} L_{-1}^j x_0 \text{ for } j = 1, \dots, n.$$

We can assume $x_0 = w$ which leads to $L_{-1}^n w = 0$. This means that $h = 0$ but then $V'_{0,\delta} \otimes L(c, 0)$ is reducible so Φ is not an isomorphism. Therefore $\gamma \notin \mathbb{Z}$.

Then from (3) it follows that $x_0 = 0$. Let $k \in \mathbb{N}$ be the smallest such that $x_k \neq 0$. Then (3) becomes

$$\sum_{i=k-1}^{n-1} w_i \otimes (i+1+\gamma)x_{i+1} = \sum_{i=k}^n w_i \otimes L_{-1}x_i$$

leading to $x_k = 0$ a contradiction.

If $\beta = 0$, the proof is essentially the same except that we consider $v_1 \otimes v$ instead of $v_0 \otimes v$. ■

Now we generalize Theorem 4.3.

Theorem 4.6 *Let $c \neq c_{p,q}$ and suppose $V(c, h)$ is reducible with degree m . Then a degree m polynomial $p(x) \in \mathbb{Q}(\alpha, \beta, h)[x]$ exists such that $V'_{\alpha, \beta} \otimes L(c, h)$ is reducible if and only if p has an integral root. For every integral root n , there is a subquotient in $V'_{\alpha, \beta} \otimes L(c, h)$ which is isomorphic to a highest weight module of the highest weight either $h - \alpha - \beta - n + 1$ or $h + 1$ in case $n = -\alpha \in \mathbb{Z}$ and $\beta = 1$.*

Proof. When $h = 0$, we have $p(x) = -(x + \alpha)$, and theorem is essentially a combination of Proposition 4.2 and Theorem 4.3. Therefore we assume $m > 1$.

First we find such polynomial p using a singular vector $uv \in V(c, h)_{h+m}$, $u \in U(\text{Vir}_-)_{-m}$. It is well known that $u = L_{-1}^m + \sum q_{i_1, \dots, i_n} L_{-i_n} \cdots L_{-i_1}$, $i_n \geq \dots \geq i_1$, $i_1 + \dots + i_n = m$ for some $q_{i_1, \dots, i_n} \in \mathbb{Q}(c, h)$ and $q_{1, \dots, 1} = 1$ (see [FF]). Then

$$u(v_{n+m-1} \otimes v) = p'(n)v_{n-1} \otimes v + \sum a_i v_{n+m-i} \otimes L_{-j_k} \cdots L_{-j_1} v$$

where $p'(n) \in \mathbb{Q}(\alpha, \beta, h)[n]$ is a degree m polynomial, $2 \leq i \leq m$, and $0 < k < m$ (since $uv = 0$). We want to eliminate the sum using induction on k . Let $\lambda = (n + m - i + \alpha + (1 - j)\beta)$. Then

$$v_{n+m-i} \otimes L_{-j} v = L_{-j}(v_{n+m-i} \otimes v) + \lambda v_{n+m-i-j} \otimes v$$

and

$$\begin{aligned} & v_{n+m-i} \otimes L_{-j_k} \cdots L_{-j_1} v = \\ & = L_{-j_k}(v_{n+m-i} \otimes L_{-j_{k-1}} \cdots L_{-j_1} v) + \lambda v_{n+m-i-j} \otimes L_{-j_{k-1}} \cdots L_{-j_1} v. \end{aligned}$$

Therefore, we get $u_i \in U(\text{Vir}_-)_{-(m-i+1)}$ such that

$$u(v_{n+m-1} \otimes v) + \sum_{i=1}^{m-1} u_i(v_{n+m-1-i} \otimes v) = p(n)v_{n-1} \otimes v \quad (4)$$

with $p(n) \in \mathbb{Q}(\alpha, \beta, h, c)[n]$ a degree m polynomial. This shows that

$$U_{n-1} \subseteq U_n + \cdots + U_{n+m-1} \quad (5)$$

if $p(n) \neq 0$.

Next we prove irreducibility in case p has no integral roots. Assume $\alpha + 2\beta \notin \mathbb{Z}$. Then

$$L_1(v_n \otimes v) = -(n + \alpha + 2\beta)v_{n+1} \otimes v \neq 0$$

so $U_n \supseteq U_{n+1}$ for all $n \in \mathbb{Z}$. Therefore (5) becomes $U_{n-1} \subseteq U_n$ and, if p has no integral roots then $U_n = U_{n+1}$ for all n . Therefore $V'_{\alpha,\beta} \otimes L(c, h)$ is irreducible. In case $\alpha = 0$ and $\beta = 0$ (resp. $\beta = 1$), U_0 (resp. U_{-1}) does not exist but $L_2(v_n \otimes v)$ shows that $U_{-1} \supseteq U_1$ (resp. $U_{-2} \supseteq U_0$). Combined with (5) this proves irreducibility.

Now suppose $\alpha + 2\beta \in \mathbb{Z}$. Since α is invariant modulo \mathbb{Z} we may assume $\alpha + 2\beta = 0$. Now we have

$$\cdots \supseteq U_{-1} \supseteq U_0, U_1 \supseteq U_2 \supseteq \cdots \quad (6)$$

If $\alpha = \beta = 0$ (resp. $\alpha = -2$ and $\beta = 1$), then U_0 (resp. U_1) does not exist so we do not have a problem. Let $\beta \neq 0, 1$ and suppose $p(2) \neq 0$. Then (5) shows that $U_1 \subseteq U_2$ which combined with (6) implies $U_1 = U_2 \subseteq U_0$. Specially, if p has no integral roots we get $U_n = U_{n+1}$ for all n , so $V'_{\alpha,\beta} \otimes L(c, h)$ is irreducible.

Now let $p(n) = 0$ for some $n \in \mathbb{Z}$. Suppose $v_{n-1} \otimes v = x(v_n \otimes v)$ for $x \in U(\text{Vir}_-)$. Note that we can write

$$v_{n-1} \otimes v = \sum_{i=0}^{k-1} x_{i+1}(v_{n+i} \otimes v)$$

for some $k \in \mathbb{N}$, and $x_j \in U(\text{Vir}_-)_{-j}$. But then $v_{n+k} \otimes x_k v = 0$ hence $x_k \in U(\text{Vir}_-)u$ which implies $k \geq m$ and $x_k = y_{k-m}u$ for some $y_{k-m} \in U(\text{Vir}_-)_{m-k}$. Suppose $k > m$. Then we can apply (4) to show that

$$x_k(v_{n+k-1} \otimes v) = y_{k-m}u(v_{n+k-1} \otimes v) = \sum_{i=1}^{m-1} y_{k-m}u_i(v_{n+k-1-i} \otimes v)$$

thus we can write $v_{n-1} \otimes v = \sum_{i=0}^{k-2} x_{i+1}(v_{n+i} \otimes v)$. Proceeding by induction, we conclude that $k = m$ and this leads back to (4). However, $p(n) = 0$ so we conclude that the equation $v_{n-1} \otimes v = x(v_n \otimes v)$ has no solution in $U(\text{Vir}_-)$. Therefore $U_{n-1} \subsetneq V'_{\alpha,\beta} \otimes L(c, h)$ and U_{n-1}/U_n is the highest weight module with highest weight $h - \alpha - \beta - n + 1$ by Lemma 4.1. In case $\alpha \in \mathbb{Z}$, $\beta = 1$ and $n = -\alpha$, there is no U_{n-1} and U_{n-2}/U_n is the highest weight module generated by highest weight $h + 1$ vector $v_{-2} \otimes v$. This completes the proof. ■

Remark 4.7 *Claims of Theorem 4.6 have been proved in Theorem 1 a) in [CGZ]. Polynomial p is closely related to φ_n in [CGZ]. Respective to terminology and definitions of the Virasoro algebra and intermediate series used in this paper, p can be regarded as a linear map $\varphi_n : U(\text{Vir}_-) \rightarrow \mathbb{C}$ defined by*

$$\varphi_n(L_{-k_r} \cdots L_{-k_1}) = \prod_{j=1}^r \left(\alpha + (1 - k_j)\beta + n + \sum_{i=1}^j k_i - 1 \right)$$

Example 4.8 If $c = \frac{10h-16h^2}{1+2h}$, a singular vector in $V(c, h)$ is $s_2 v = (L_{-1}^2 - \frac{4h+2}{3} L_{-2})v$. By direct computation we get

$$\begin{aligned} & \left(\frac{-1}{n+\alpha+2\beta} s_2 L_1 + 2(n+1+\alpha) L_{-1} \right) (v_n \otimes v) = \\ & = - \left((n+1+\alpha)(n+\alpha) - \frac{4h+2}{3} (n+1+\alpha-\beta) \right) v_{n-1} \otimes v. \end{aligned} \quad (7)$$

The roots of $p(n) = (n+1+\alpha)(n+\alpha) - \frac{4h+2}{3} (n+1+\alpha-\beta)$ are $-\alpha + \frac{4h-1}{6} \pm \frac{1}{6} \sqrt{(4h+5)^2 - 24\beta(2h+1)}$.

Example 4.9 For $m = 3$, the roots of polynomial p are $-\alpha + h$ and $-\alpha + \frac{h-1}{2} \pm \frac{1}{2} \sqrt{(h+3)^2 - 8\beta(h+1)}$.

Corollary 4.10 Suppose Verma module $V(c, h)$ contains a weight 2 singular vector and $-\alpha + \frac{4h-1 \pm \sqrt{(4h+5)^2 - 24\beta(2h+1)}}{6} \notin \mathbb{Z}$. Then $V'_{\alpha, \beta} \otimes L(c, h)$ is irreducible.

Corollary 4.11 If Verma module $V(c, h)$ has a weight 3 singular vector and $-\alpha + \frac{h-1 \pm \sqrt{(h+3)^2 - 8\beta(h+1)}}{2}, -\alpha + h \notin \mathbb{Z}$, module $V'_{\alpha, \beta} \otimes L(c, h)$ is irreducible.

Remark 4.12 When $c = c_{p,q}$, reducible Verma module $V(c, h)$ has two independent singular vectors. Each of them produces two polynomials as in Theorem 4.6. $V'_{\alpha, \beta} \otimes L(c, h)$ is irreducible unless there is a common integral root. Proving reducibility directly, like in Theorem 4.6 seems somewhat challenging. However, this will follow from the existence of intertwining operators, as we will see in the following section.

5 Intertwining operators and reducibility of $V'_{\alpha, \beta} \otimes L(c, h)$

Vertex operator algebras (VOAs) are a fundamental class of algebraic structures which have arisen in mathematics and physics a few decades ago. Their importance is supported by their numerous relations with many fields of algebra, representation theory, topology, differential equations and conformal field theory. The main original motivation for the introduction of the notion of VOA arose from the problem of realizing the monster sporadic group as a symmetry group of a certain infinite-dimensional vector space ([FLM]).

An interested reader should consult [FHL] or [LL] for a detailed approach. Here we present only basic definitions which should suffice our needs.

For any algebraic expression z we set $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$ provided that this sum makes sense. This is the formal analogue of the δ -distribution at $z = 1$; in particular, $\delta(z)f(z) = \delta(z)f(1)$ for any f for which these expressions are defined.

Definition 1 A *vertex operator algebra* $(V, Y, \mathbf{1})$ is a \mathbb{Z} -graded vector space (graded by weights) $V = \bigoplus_{n \in \mathbb{Z}} V_{(n)}$ such that $\dim V_{(n)} < \infty$ for $n \in \mathbb{Z}$ and $V_{(n)} = 0$ for n sufficiently small, equipped with a linear map $V \otimes V \rightarrow V[[z, z^{-1}]]$, or equivalently,

$$V \rightarrow (\text{End } V)[[z, z^{-1}]]$$

$$v \mapsto Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \quad (\text{where } v_n \in \text{End } V),$$

$Y(v, z)$ denoting the vertex operator associated with v , and equipped with two distinguished homogeneous vectors $\mathbf{1}$ (the vacuum) and $\omega \in V$. The following conditions are assumed for $u, v \in V$:

$$u_n v = 0 \text{ for } n \text{ sufficiently large;}$$

$$Y(\mathbf{1}, z) = \mathbf{1} (= \text{id}_V);$$

the creation property holds:

$$Y(v, z)\mathbf{1} \in V[[z]] \quad \text{and} \quad \lim_{z \rightarrow 0} Y(v, z)\mathbf{1} = 0$$

(that is, $Y(v, z)\mathbf{1}$ involves only nonnegative integral powers of z and the constant term is v); the Jacobi identity:

$$\begin{aligned} & z_0^{-1} \delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u, z_1) Y(v, z_2) - z_0^{-1} \delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(v, z_2) Y(u, z_1) \\ &= z_2^{-1} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u, z_0)v, z_2); \end{aligned} \quad (8)$$

the Virasoro algebra relations:

$$[L_m, L_n] = (m - n) L_{m+n} + \frac{m^3 - m}{12} \delta_{m+n, 0} (\text{rank } V)$$

for $m, n \in \mathbb{Z}$, where

$$L_n = \omega_{n+1} \text{ for } n \in \mathbb{Z}, \text{ i.e. } Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$$

and $\text{rank } V \in \mathbb{C}$, $L_0 v = n v = (\text{wt } v) v$ for $n \in \mathbb{Z}$ and $v \in V_{(n)}$;

$$\frac{d}{dz} Y(v, z) = Y(L_{-1} v, z)$$

(the L_{-1} -derivative property).

Definition 2 Given a VOA $(V, Y, \mathbf{1})$, a *module* (W, \mathcal{Y}) for V is a \mathbb{Q} -graded vector space $W = \bigoplus_{n \in \mathbb{Q}} W_{(n)}$ such that $\dim W_{(n)} < \infty$ for $n \in \mathbb{Q}$, and $W_{(n)} = 0$

for n sufficiently small, equipped with a linear map $V \otimes W \rightarrow W [[z, z^{-1}]]$, or equivalently,

$$\begin{aligned} V &\rightarrow (\text{End } W) [[z, z^{-1}]] \\ v &\mapsto \mathcal{Y}(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \text{ (where } v_n \in \text{End } W), \end{aligned}$$

$\mathcal{Y}(v, z)$ denoting the vertex operator associated with v . The Virasoro algebra relations hold on W with scalar equal to $\text{rank } V$:

$$[L_m, L_n] = (m - n) L_{m+n} + \frac{m^3 - m}{12} \delta_{m+n, 0} (\text{rank } V)$$

for $m, n \in \mathbb{Z}$, where

$$L_n = \omega_{n+1} \text{ for } n \in \mathbb{Z}, \text{ i.e. } \mathcal{Y}(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$$

$L_0 w = n w$ for $n \in \mathbb{Q}$ and $w \in W_{(n)}$. For $u, v \in V$ and $w \in W$ the following properties hold:

- (i) Truncation property: $v_n w = 0$ for n sufficiently large and $Y(\mathbf{1}, z) = 1$;
- (ii) L_{-1} -derivative property

$$\frac{d}{dz} \mathcal{Y}(v, z) = \mathcal{Y}(L_{-1} v, z)$$

- (iii) The Jacobi identity

$$\begin{aligned} & z_0^{-1} \delta \left(\frac{z_1 - z_2}{z_0} \right) \mathcal{Y}(u, z_1) \mathcal{Y}(v, z_2) - z_0^{-1} \delta \left(\frac{z_2 - z_1}{-z_0} \right) \mathcal{Y}(v, z_2) \mathcal{Y}(u, z_1) \\ &= z_2^{-1} \delta \left(\frac{z_1 - z_0}{z_2} \right) \mathcal{Y}(Y(u, z_0) v, z_2); \end{aligned}$$

($Y(u, z_0)$ is the operator associated with V).

Definition 3 Let $V = (V, Y, \mathbf{1})$ be a vertex operator algebra, and (W_i, Y_i) , $i = 1, 2, 3$ three (not necessarily distinct) V -modules. **Intertwining operator** of type $\binom{W_3}{W_1 \ W_2}$ is a linear map $W_1 \otimes W_2 \rightarrow W_3 \{z\} = \left\{ \sum_{n \in \mathbb{Q}} u_n z^n : u_n \in W_k \right\}$, or equivalently,

$$\begin{aligned} W_1 &\rightarrow (\text{Hom}(W_2, W_3)) \{z\} \\ w &\mapsto \mathcal{I}(w, z) = \sum_{n \in \mathbb{Q}} w_n z^{-n-1}, \text{ with } w_n \in \text{Hom}(W_2, W_3). \end{aligned}$$

For any $v \in V$, $u \in W_1$, $w \in W_2$ the following conditions are satisfied:

- (i) *Truncation property* - $u_nv = 0$ for n sufficiently large;
- (ii) *L_{-1} -derivative property* - $\mathcal{I}(L_{-1}u, z) = \frac{d}{dz}\mathcal{I}(u, z)$;
- (iii) *The Jacobi identity*

$$\begin{aligned} & z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)Y_3(v, z_1)\mathcal{I}(u, z_2)w - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right)\mathcal{I}(u, z_2)Y_2(v, z_1)w \\ &= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\mathcal{I}(Y_1(v, z_0)u, z_2)w. \end{aligned}$$

Now, if M_i , $i = 1, 2, 3$ are highest weight Vir-modules of highest weights h_i and central charge c , and \mathcal{I} is an intertwining operator of type $\begin{pmatrix} M_3 \\ M_1 \quad M_2 \end{pmatrix}$, then $\mathcal{I}(u, z) = z^{-\alpha} \sum_{n \in \mathbb{Z}} u_n z^{-n-1}$ where $\alpha = h_1 + h_2 - h_3$ [FZ]. Suppose there exists such a nontrivial intertwining operator \mathcal{I} and suppose $h_1 \neq 0$. Let v the highest weight vector in M_1 . Equating the coefficient of $z_0^{-1}z_1^{-m-1}z_2^{-n-1}$ in (8) yields

$$[u_m, v_n] = \sum_{i \geq 0} \binom{m}{i} (u_i v)_{m+n-i},$$

and in particular for $u = \omega$ we have

$$\begin{aligned} [L_m, v_n] &= \sum_{i \geq 0} \binom{m+1}{i} (L_{i-1}v)_{m+n-i+1} = \\ &= (L_{-1}v)_{m+n+1} + (m+1)(L_0v)_{m+n} = \\ &= -(\alpha + n + m + 1)v_{m+n} + (m+1)h_1v_{m+n} = \\ &= -(n + \alpha + (1 - h_1)(1 + m))v_{m+n}, \end{aligned}$$

hence the components of $\mathcal{I}(v, z)$ span $V'_{\alpha, \beta}$, where $\beta = 1 - h_1$. Moreover, a nontrivial Vir-homomorphism Φ is defined:

$$\Phi : V'_{\alpha, \beta} \otimes M_2 \rightarrow M_3, \quad \Phi(v_n \otimes v') = v_n v'$$

where v' is the highest weight vector in M_2 . Comparing the dimensions of the weight subspaces we conclude that $V'_{\alpha, \beta} \otimes M_2$ is reducible.

Theorem 5.1 *Let M_i , $i = 1, 2, 3$ be the highest weight Vir-modules with the highest weight (c, h_i) , and $h_1 \neq 0$. Suppose a nontrivial intertwining operator \mathcal{I} of type $\begin{pmatrix} M_3 \\ M_1 \quad M_2 \end{pmatrix}$ exists. Then there exists nontrivial Vir-homomorphism $V'_{\alpha, \beta} \otimes M_2 \rightarrow M_3$, where $\alpha = h_1 + h_2 - h_3$ and $\beta = 1 - h_1$. Consequently, $V'_{\alpha, \beta} \otimes M_2$ is reducible.*

5.1 Minimal models

Vir-module $L(c, 0)$ admits an irreducible VOA structure. For $c = c_{p,q} = 1 - 6\frac{(p-q)^2}{pq}$, where $p, q > 1$ are relatively prime, this algebra is rational ([FZ]) i.e. it has only finitely many irreducible modules and every finitely generated module is a direct sum of irreducibles. Let

$$h_{m,n} = \frac{(np - mq)^2 - (p - q)^2}{4pq}, 0 < m < p, 0 < n < q. \quad (9)$$

Then $L(c_{p,q}, h_{m,n})$ is a module over VOA $L(c_{p,q}, 0)$ called a **minimal model**. Wang has shown in [W] that minimal models are all irreducibles for VOA $L(c_{p,q}, 0)$.

Using fusion rules ([FZ], [W]) we have the complete list of intertwining operators for VOA $L(c_{p,q}, 0)$.

An ordered triple of pairs of integers $((m, n), (m', n'), (m'', n''))$ is **admissible** if $0 < m_1, m_2, m_3 < p$, $0 < n_1, n_2, n_3 < q$, $m_1 + m_2 + m_3 < 2p$, $n_1 + n_2 + n_3 < 2q$, $m_1 < m_2 + m_3$, $m_2 < m_1 + m_3$, $m_3 < m_1 + m_2$, $n_1 < n_2 + n_3$, $n_2 < n_1 + n_3$, $n_3 < n_1 + n_2$ and the sums $m_1 + m_2 + m_3$ and $n_1 + n_2 + n_3$ are odd. We identify the triples $((m_1, n_1), (m_2, n_2), (m_3, n_3))$ and $((m_1, n_1), (p - m_2, q - n_2), (p - m_3, q - n_3))$.

Let $c = c_{p,q}$. A nontrivial intertwining operator of type $\begin{pmatrix} L(c, h_{m'', n''}) \\ L(c, h_{m, n}) & L(c, h_{m', n'}) \end{pmatrix}$ exists if and only if $((m, n), (m', n'), (m'', n''))$ is an admissible triple of pairs. (See [W]).

If $h = h_{m,n}$ as in (9), the following intertwining operators always exist:

$$\begin{aligned} & \begin{pmatrix} L(c, h) \\ L(c, 0) & L(c, h) \end{pmatrix} \text{ (module } L(c, h)) \\ & \begin{pmatrix} L(c, h) \\ L(c, h) & L(c, 0) \end{pmatrix} \text{ (transposed operator)} \\ & \begin{pmatrix} L(c, 0) \\ L(c, h) & L(c, h) \end{pmatrix} \text{ (adjoint operator)} \end{aligned}$$

Example 5.2 (Yang-Lee model) $L(-\frac{22}{5}, 0)$ and $L(-\frac{22}{5}, -\frac{1}{5})$ are the only irreducibles over VOA $L(-\frac{22}{5}, 0)$. Operator of type $\begin{pmatrix} L(-\frac{22}{5}, -\frac{1}{5}) \\ L(-\frac{22}{5}, -\frac{1}{5}) & L(-\frac{22}{5}, -\frac{1}{5}) \end{pmatrix}$ completes the list of intertwining operators for $L(-\frac{22}{5}, 0)$.

Example 5.3 (Ising model) $L(\frac{1}{2}, 0)$, $L(\frac{1}{2}, \frac{1}{16})$ and $L(\frac{1}{2}, \frac{1}{2})$ are the only irreducibles over $L(\frac{1}{2}, 0)$. Operators of type $\begin{pmatrix} L(\frac{1}{2}, \frac{1}{16}) \\ L(\frac{1}{2}, \frac{1}{2}) & L(\frac{1}{2}, \frac{1}{16}) \end{pmatrix}$, $\begin{pmatrix} L(\frac{1}{2}, \frac{1}{16}) \\ L(\frac{1}{2}, \frac{1}{16}) & L(\frac{1}{2}, \frac{1}{2}) \end{pmatrix}$ and $\begin{pmatrix} L(\frac{1}{2}, \frac{1}{2}) \\ L(\frac{1}{2}, \frac{1}{16}) & L(\frac{1}{2}, \frac{1}{16}) \end{pmatrix}$ complete the list of intertwining operators for $c = \frac{1}{2}$.

Corollary 5.4 There are nontrivial Vir-homomorphisms:

$$\begin{aligned} & \left(V'_{0, \frac{6}{5}} \otimes L(-\frac{22}{5}, 0) \right) \rightarrow L(-\frac{22}{5}, -\frac{1}{5}), \\ & \left(V'_{-\frac{2}{5}, \frac{6}{5}} \otimes L(-\frac{22}{5}, -\frac{1}{5}) \right) \rightarrow L(-\frac{22}{5}, 0), \quad \left(V'_{-\frac{1}{5}, \frac{6}{5}} \otimes L(-\frac{22}{5}, -\frac{1}{5}) \right) \rightarrow L(-\frac{22}{5}, -\frac{1}{5}); \end{aligned}$$

$$\begin{aligned}
& \left(V'_{0, \frac{1}{2}} \otimes L(\frac{1}{2}, 0) \right) \rightarrow L(\frac{1}{2}, \frac{1}{2}), \quad \left(V'_{0, \frac{15}{16}} \otimes L(\frac{1}{2}, 0) \right) \rightarrow L(\frac{1}{2}, \frac{1}{16}), \\
& \left(V'_{0, \frac{1}{2}} \otimes L(\frac{1}{2}, \frac{1}{2}) \right) \rightarrow L(\frac{1}{2}, 0), \quad \left(V'_{\frac{1}{2}, \frac{15}{16}} \otimes L(\frac{1}{2}, \frac{1}{2}) \right) \rightarrow L(\frac{1}{2}, \frac{1}{16}), \\
& \left(V'_{\frac{1}{8}, \frac{15}{16}} \otimes L(\frac{1}{2}, \frac{1}{16}) \right) \rightarrow L(\frac{1}{2}, 0), \quad \left(V'_{\frac{1}{2}, \frac{1}{2}} \otimes L(\frac{1}{2}, \frac{1}{16}) \right) \rightarrow L(\frac{1}{2}, \frac{1}{16}), \\
& \left(V'_{-\frac{3}{8}, \frac{15}{16}} \otimes L(\frac{1}{2}, \frac{1}{16}) \right) \rightarrow L(\frac{1}{2}, \frac{1}{2}).
\end{aligned}$$

Proof. Directly from Theorem 5.1. ■

Remark 5.5 Let $c \neq c_{p,q}$ and $h \neq 0$. Then $V(c, h)$ is a module over VOA $L(c, 0)$ so there exists a transposed intertwining operator of type $\left(\begin{smallmatrix} V(c, h) \\ V(c, h) \end{smallmatrix} \right)_{L(c, 0)}$. Therefore, a nontrivial Vir-epimorphism $V'_{\alpha, \beta} \otimes L(c, 0) \rightarrow V(c, h)$ exists which proves Theorem 4.3. However, $V(c_{p,q}, h)$ is not $L(c_{p,q}, 0)$ -module.

Next we consider irreducibility of $V'_{\alpha, \beta} \otimes L(c_{p,q}, h_{m,n})$, where $c = -\frac{22}{5}, \frac{1}{2}$. We take advantage of the fact that $V(c_{p,q}, h_{m,n})$ has two singular vectors that we can use independently. This way we prove irreducibility for all pairs (α, β) except those laying on intersection of two algebraic curves. These exceptions are precisely those listed in Corollary 5.4.

Proposition 5.6 Module $V'_{\alpha, \beta} \otimes L(-\frac{22}{5}, 0)$ is irreducible if and only if $(\alpha, \beta) \neq (0, \frac{6}{5})$. Moreover,

$$\left(V'_{0, \frac{6}{5}} \otimes L(-\frac{22}{5}, 0) \right) / U_0 \cong L(-\frac{22}{5}, -\frac{1}{5}).$$

Proof. If $\alpha \notin \mathbb{Z}$, this is a special case of Proposition 4.2. Let $\alpha = 0$ and let $s = L^2_{-2} - \frac{3}{5}L_{-4}$. Then $sv = 0$ if v is the highest weight vector in $L(-\frac{22}{5}, 0)$. In the discussion preceeding Theorem 4.3 we noted that

$$\cdots = U_{-2} = U_{-1} \supseteq U_0 = U_1 = \cdots$$

It remains to show that $U_{-1} \subseteq U_0$ (or $U_{-2} \subseteq U_1$ if $\beta = 0, 1$). We have

$$s(v_3 \otimes v) + 2(3 - \beta)L_{-2}(v_1 \otimes v) = \left(\beta - \frac{6}{5} \right) (1 - \beta)v_{-1} \otimes v.$$

Since $v_3 \otimes v \in U_3 = U_1$ we obtain $U_1 \subseteq U_{-1}$ for $\beta \neq 1, \frac{6}{5}$. On the other hand, for $\beta \neq 0$ we have

$$s(v_2 \otimes v) - 2(\beta - 2)L_{-2}(v_0 \otimes v) = \left(\frac{6}{5} - \beta \right) (\beta + 1)v_{-2} \otimes v,$$

proving that $U_0 \subseteq U_{-2}$, for $\beta \neq -1, 0, \frac{6}{5}$. This proves irreducibility when $\beta \neq \frac{6}{5}$.

Since $V'_{0, \frac{6}{5}} \otimes L(-\frac{22}{5}, 0)$ is reducible by Corollary 5.4, then by Theorem 3.2 we must have $U_0 \neq U_{-1}$. From lema 4.1 we know that $\left(V'_{0, \frac{6}{5}} \otimes L(-\frac{22}{5}, 0) \right) / U_0$

is isomorphic to some quotient of $V(-\frac{22}{5}, -\frac{1}{5})$. Next we check relations for singular vectors in $V(-\frac{22}{5}, -\frac{1}{5})$. We need to show that

$$(L_{-1}^2 - \frac{2}{5}L_{-2})(v_{-1} \otimes v) \in U_0$$

$$(L_{-1}^3 - \frac{8}{5}L_{-2}L_{-1} - \frac{4}{25}L_{-3})(v_{-1} \otimes v) \in U_0$$

By direct computation we see that

$$(L_{-1}^2 - \frac{2}{5}L_{-2})(v_{-1} \otimes v) = -s(v_1 \otimes v) \in U_1 = U_0$$

and

$$\begin{aligned} & (L_{-1}^3 - \frac{8}{5}L_{-2}L_{-1} - \frac{4}{25}L_{-3})(v_{-1} \otimes v) = \\ & = -\frac{75}{2}(L_{-1}^2 - \frac{2}{5}L_{-2})(v_0 \otimes v) - 25L_{-1}(L_{-1}^2 - \frac{2}{5}L_{-2})(v_1 \otimes v) \in U_0 \end{aligned}$$

This shows irreducibility of $(V'_{0, \frac{6}{5}} \otimes L(-\frac{22}{5}, 0)) / U_0$ and completes the proof. ■

Proposition 5.7 $V'_{\alpha, \beta} \otimes L(\frac{1}{2}, 0)$ is irreducible if and only if $(\alpha, \beta) \neq (0, \frac{1}{2}), (0, \frac{15}{16})$.
Moreover

$$\begin{aligned} & \left(V'_{0, \frac{1}{2}} \otimes L(\frac{1}{2}, 0) \right) / U_0 \cong L(\frac{1}{2}, \frac{1}{2}), \\ & \left(V'_{0, \frac{15}{16}} \otimes L(\frac{1}{2}, 0) \right) / U_0 \cong L(\frac{1}{2}, \frac{1}{16}). \end{aligned}$$

Proof. Again, we may assume $\alpha = 0$, and need to check that $U_{-2} \subseteq U_1$.

Let $s' = 64L_{-2}^3 + 93L_{-3}^2 - 264L_{-4}L_{-2} - 108L_{-6}$. Then $s'v = 0$ in $L(\frac{1}{2}, 0)$. The relations

$$\begin{aligned} & s'(v_5 \otimes v) + 192(5 - \beta)L_{-2}^2(v_3 \otimes v) - 264(5 - \beta)L_{-4}(v_3 \otimes v) + \\ & + 186(5 - 2\beta)L_{-3}(v_2 \otimes v) - 264(5 - 3\beta)L_{-2}(v_1 \otimes v) + \\ & + 192(5 - \beta)(3 - \beta)L_{-2}(v_1 \otimes v) = -2(\beta - 1) \left(\beta - \frac{1}{2} \right) \left(\beta - \frac{15}{16} \right) (v_{-1} \otimes v) \end{aligned}$$

and

$$\begin{aligned} & s'(v_4 \otimes v) + 192(4 - \beta)L_{-2}^2(v_2 \otimes v) - 264(4 - \beta)L_{-4}(v_2 \otimes v) + \\ & + 186(4 - 2\beta)L_{-3}(v_1 \otimes v) - 264(4 - 3\beta)L_{-1}(v_0 \otimes v) + \\ & + 192(4 - \beta)(2 - \beta)L_{-1}(v_0 \otimes v) = 2(\beta + 2) \left(\beta - \frac{1}{2} \right) \left(\beta - \frac{15}{16} \right) (v_{-2} \otimes v) \end{aligned}$$

show that $U_1 \subseteq U_{-2}$ when $\beta \neq \frac{1}{2}, \frac{15}{16}$.

Let $\beta = \frac{1}{2}$. Using L_1 , L_2 and L_{-1} we get

$$\left(V'_{0, \frac{1}{2}} \otimes L\left(\frac{1}{2}, 0\right) \right) = U_{-1} \supseteq U_0 = U_1 = \dots$$

Since $V'_{0, \frac{1}{2}} \otimes L(\frac{1}{2}, 0)$ is reducible by Corollary 5.4, we must have $U_0 \neq U_1$. By Lemma 4.1, U_{-1}/U_0 is isomorphic to some quotient of $V(\frac{1}{2}, \frac{1}{2})$. It remains to show that

$$\begin{aligned} (L_{-1}^2 - \frac{4}{3}L_{-2})(v_{-1} \otimes v) &\in U_0, \\ (L_{-3} - \frac{4}{5}L_{-2}L_{-1})(v_{-1} \otimes v) &\in U_0. \end{aligned}$$

However, by direct computation we find

$$\begin{aligned} (L_{-1}^2 - \frac{4}{3}L_{-2})(v_{-1} \otimes v) &= -\frac{4}{3}v_{-1} \otimes L_{-2}v = s_6(v_3 \otimes v) + \\ &+ 480L_{-2}^2(v_1 \otimes v) + 372L_{-3}(v_0 \otimes v) - 660L_{-4}(v_1 \otimes v) \in U_0 \end{aligned}$$

where s_6v is a weight 6 singular vector in $V(\frac{1}{2}, 0)$. Also, $(L_{-3} - \frac{4}{5}L_{-2}L_{-1})(v_{-1} \otimes v) = 0$.

If $\beta = \frac{15}{16}$, in a similar fashion one can show that

$$\begin{aligned} (L_{-1}^2 - \frac{3}{4}L_{-2})(v_{-1} \otimes v) &\in U_0, \\ (16L_{-2}^2 - 24L_{-3}L_{-1} - 9L_{-4})(v_{-1} \otimes v) &\in U_0. \end{aligned}$$

■

When $h_{m,n} \neq 0$ we proceed similarly. Using L_1 , L_2 and singular vectors on levels 2, 3 or 4 one can prove

Proposition 5.8 $V'_{\alpha, \beta} \otimes L(-\frac{22}{5}, -\frac{1}{5})$ is irreducible if and only if $(\alpha, \beta) \neq (-\frac{2}{5}, \frac{6}{5}), (-\frac{1}{5}, \frac{6}{5})$.

$V'_{\alpha, \beta} \otimes L(\frac{1}{2}, \frac{1}{2})$ is irreducible if and only if $(\alpha, \beta) \neq (0, \frac{1}{2}), (\frac{1}{2}, \frac{15}{16})$.

$V'_{\alpha, \beta} \otimes L(\frac{1}{2}, \frac{1}{16})$ is irreducible if and only if $(\alpha, \beta) \neq (\frac{1}{8}, \frac{15}{16}), (-\frac{3}{8}, \frac{15}{16}), (\frac{1}{2}, \frac{1}{2})$.

Remark 5.9 By direct computation, just as in Propositions 5.6-5.7, one can show that the kernel of each homomorphism in Corollary 5.4 is U_k for some $k \in \{-2, -1, 0\}$. Every such U_k is irreducible (by Corollary 3.3). Therefore $V'_{\alpha, \beta} \otimes L(c_{p,q}, h_{m,n})$ has a Jordan-Hölder composition of length 2.

Based on the examples shown above and other examples not mentioned here (such as $c_{2,7} = -68/7$ minimal models) we state

Conjecture 5.10 Let $c = c_{p,q} \neq 0$ and let $L(c, h)$ be a minimal model. The module $V'_{\alpha, \beta} \otimes L(c, h)$ is reducible if and only if there exists an admissible triple $((m, n), (m', n'), (m'', n''))$ such that $h = h_{m', n'}$, $\alpha = h_{m, n} + h_{m', n'} - h_{m'', n''}$ and $\beta = 1 - h_{m, n}$. In this case there is an irreducible submodule U such that

$$(V'_{\alpha, \beta} \otimes L(c, h)) / U \cong L(c, h_{m'', n''}).$$

5.2 c=1 intertwining operators

It is known (see [KR], [FZ]) that $V(1, h) = L(1, h)$ if and only if $h \neq \frac{m^2}{4}$ for $m \in \mathbb{Z}$. In case $h = m^2$, the unique maximal submodule of $V(1, m^2)$ is generated by a weight $(m+1)^2$ vector and is isomorphic to $V(1, (m+1)^2)$. We have the following fusion rules for VOA $L(1, 0)$ from [DJ], [Mi]:

Let $m, n, k \in \mathbb{N}$. An intertwining operator of type $\left(\begin{smallmatrix} L(1, k^2) \\ L(1, m^2) & L(1, n^2) \end{smallmatrix} \right)$ exists if and only if $|n - m| \leq k \leq n + m$. If $n \neq p^2$ for all $p \in \mathbb{Z}$, then the operator of type $\left(\begin{smallmatrix} L(1, k) \\ L(1, m^2) & L(1, n) \end{smallmatrix} \right)$ exists if and only if $k = n$. Also, a transposed operator of type $\left(\begin{smallmatrix} L(1, k) \\ L(1, n) & L(1, m^2) \end{smallmatrix} \right)$ exists if and only if $k = n$.

Now consider $L(1, 1)$. For a fixed $m \in \mathbb{N}$ we have the operators of types $\left(\begin{smallmatrix} L(1, (m-1)^2) \\ L(1, m^2) & L(1, 1) \end{smallmatrix} \right)$, $\left(\begin{smallmatrix} L(1, m^2) \\ L(1, m^2) & L(1, 1) \end{smallmatrix} \right)$ and $\left(\begin{smallmatrix} L(1, (m+1)^2) \\ L(1, m^2) & L(1, 1) \end{smallmatrix} \right)$ when $n \neq p^2$. Since $\alpha = m^2 + 1 - k^2 \in \mathbb{Z}$, this proves reducibility of $V'_{0, 1-m^2} \otimes L(1, 1)$. From the existence of a transposed operator of type $\left(\begin{smallmatrix} L(1, n) \\ L(1, n) & L(1, 1) \end{smallmatrix} \right)$, we get reducibility of $V'_{0, 1-n} \otimes L(1, 1)$ for all $n \in \mathbb{N}$.

Example 5.11 Now we apply Theorem 4.6 to $V'_{\alpha, \beta} \otimes L(1, 1)$. The Verma module $V(1, 1)$ is reducible with degree 3, and the associated polynomial has roots $-\alpha \pm 2\sqrt{1-\beta}$ and $1-\alpha$ (see Example 4.9). If we set $\alpha = 0$, then U_0/U_1 is the highest weight module of the highest weight $1-\beta$. If $2\sqrt{1-\beta} \in \mathbb{Z}$, or equivalently if $1-\beta = \frac{m^2}{4}$ for some $m \in \mathbb{Z}$, then U_{m-1}/U_m is the highest weight module of the weight $(\frac{m}{2}+1)^2$ and U_{-m-1}/U_{-m} is the highest weight module of the weight $(\frac{m}{2}-1)^2$. This indicates existence of the intertwining operators of types

$$\left(\begin{smallmatrix} L(1, h) \\ L(1, h) & L(1, 1) \end{smallmatrix} \right), \left(\begin{smallmatrix} L(1, (\frac{m+2}{2})^2) \\ L(1, (\frac{m}{2})^2) & L(1, 1) \end{smallmatrix} \right), \left(\begin{smallmatrix} L(1, (\frac{m-2}{2})^2) \\ L(1, (\frac{m}{2})^2) & L(1, 1) \end{smallmatrix} \right),$$

for $h \in \mathbb{C}$, and $m \in \mathbb{Z}$ which agrees with (specially if $h = \frac{m^2}{4}$) results from [DJ].

Of course, for every $\alpha \in \mathbb{C} \setminus \mathbb{Z}$ there are two nontrivial subquotients, namely $U_{-\alpha \pm 2\sqrt{1-\beta}-1}/U_{-\alpha \pm 2\sqrt{1-\beta}}$, but these are isomorphic to irreducible Verma modules with highest weights $(\sqrt{1-\beta} \mp 1)^2 \notin \frac{1}{4}\mathbb{Z}^2$. This could lead to the existence of the operators of type $\left(\begin{smallmatrix} L(1, (\sqrt{h}-1)^2) \\ L(1, h) & L(1, 1) \end{smallmatrix} \right)$ and $\left(\begin{smallmatrix} L(1, (\sqrt{h}+1)^2) \\ L(1, h) & L(1, 1) \end{smallmatrix} \right)$.

Corollary 5.12 If an intertwining operator of type $\left(\begin{smallmatrix} L(1, h') \\ L(1, h) & L(1, 1) \end{smallmatrix} \right)$ exists, then $h' \in \left\{ (\sqrt{h}-1)^2, h, (\sqrt{h}+1)^2 \right\}$. In particular if an intertwining operator of type $\left(\begin{smallmatrix} L(1, (\frac{k}{2})^2) \\ L(1, (\frac{m}{2})^2) & L(1, 1) \end{smallmatrix} \right)$ exists for some $m, k \in \mathbb{Z}_+$, then $k \in \{m-2, m, m+2\}$.

Proof. Directly from Theorem 5.1 and the previous example. ■

Example 5.13 We can also apply Theorem 4.6 to $V'_{\alpha,\beta} \otimes L(1, \frac{1}{4})$. There is a singular vector in $V(1, \frac{1}{4})$ at level 2, so the roots of the associated polynomial are $-\alpha \pm \sqrt{1-\beta}$. If we want $1-\beta = \frac{m^2}{4}$, then $\alpha = \frac{k}{2}$ for some $k \in \mathbb{Z}$. In this case, $U_{\frac{\pm m-k}{2}-1}/U_{\frac{\pm m-k}{2}}$ is the highest weight module of the weight $\frac{(m \mp 1)^2}{4}$. This gives the operators of type

$$\begin{pmatrix} L(1, (\frac{m-1}{2})^2) \\ L(1, (\frac{m}{2})^2) & L(1, \frac{1}{4}) \end{pmatrix} \text{ and } \begin{pmatrix} L(1, (\frac{m+1}{2})^2) \\ L(1, (\frac{m}{2})^2) & L(1, \frac{1}{4}) \end{pmatrix}.$$

Existence of these intertwining operators was proved in [Mi]. For $\alpha \notin \frac{1}{2}\mathbb{Z}$ we get a subquotient isomorphic to a highest weight module of the weight $(\sqrt{1-\beta} \mp \frac{1}{2})^2$ indicating operators of type $\begin{pmatrix} L(1, (\sqrt{h}-\frac{1}{2})^2) \\ L(1,h) & L(1,1) \end{pmatrix}$ and $\begin{pmatrix} L(1, (\sqrt{h}+\frac{1}{2})^2) \\ L(1,h) & L(1,1) \end{pmatrix}$.

Corollary 5.14 If an intertwining operator of type $\begin{pmatrix} L(1,h') \\ L(1,h) & L(1,\frac{1}{4}) \end{pmatrix}$ exists, then $h' = (\sqrt{h} \pm \frac{1}{2})^2$. In particular, let $m, k \in \mathbb{Z}_+$. If an intertwining operator of type $\begin{pmatrix} L(1, (\frac{k}{2})^2) \\ L(1, (\frac{m}{2})^2) & L(1, \frac{1}{4}) \end{pmatrix}$ exists, then $k = m \pm 1$.

Remark 5.15 Let $\varkappa \notin \mathbb{Q}$, $c = 13 - 6\varkappa - 6\varkappa^{-1}$ and $\Delta(k) = \frac{k(k+2)}{4\varkappa} - \frac{k}{2}$ for some $k \in \mathbb{Z}$. Then $V(c, \Delta(k))$ is irreducible if and only if $k < 0$ (see [FF2]). Furthermore, if $k_i \in \mathbb{N}$, $i = 1, 2, 3$ then an intertwining operator of type $\begin{pmatrix} L(c, \Delta(k_3)) \\ L(c, \Delta(k_1)) & L(c, \Delta(k_2)) \end{pmatrix}$ exists if and only if $k_1 + k_2 + k_3 \in 2\mathbb{Z}$ and $|k_1 - k_2| \leq k_3 \leq k_1 + k_2$ (see Proposition 2.24. in [FZ2]). However, if we allow $\varkappa = 1$, then $c = 1$ and $\Delta(k) = \frac{k^2}{4}$. In this case it seems that these results for the intertwining operators still hold.

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